Quantum Thermal Effect of Dirac Particles in a Non-uniformly Rectilinearly Accelerating Kinnersley Black Hole

WU Shuang-Qing* and CAI Xu†
Institute of Particle Physics, Hua-Zhong Normal University, Wuhan 430079, China
*E-mail: sqwu@iopp.ccnu.edu.cn †E-mail: xcai@ccnu.edu.cn

(July 7, 2001)

The Hawking radiation of Dirac particles in an arbitrarily rectilinearly accelerating Kinnersley black hole is studied by using a method of the generalized tortoise coordinate transformation. Both the location and the temperature of the event horizon depend on the time and polar angle. The Hawking thermal radiation spectrum of Dirac particles is also derived.

PACS numbers: 97.60.Lf, 04.70.Dy

Last decades has witnessed much progress in investigating the Hawking evaporation [1] of scalar fields or Dirac particles in the stationary axisymmetry black holes. [2,3] To study the Hawking radiation of a non-stationary black hole, Zhao and Dai [4] proposed an effective method called by the generalized tortoise coordinate transformation (GTCT). This method has been applied to discuss quantum thermal effect of scalar particles in some non-uniformly accelerating black holes [5] and that of Dirac particles in the non-static black hole. [6] However, it is very difficult to study the quantum thermal effect of Dirac particles in a non-spherically symmetric and non-stationary black hole. The difficulty is due to the non-separability of the Chandrasekhar-Dirac equation [7] in the most general space-times.

Recently we succeed in dealing with the Hawking radiation of Dirac particles in a variable-mass Kerr black hole. [8] In this letter, we extend this treatment to the Hawking effect of Dirac particles in a non-uniformly rectilinearly accelerating Kinnersley black hole. [9] With the aid of the GTCT method, we study the asymptotic behaviors of first-order and second-order forms of Dirac equation near the event horizon. We present the equation that determines the location of the event horizon of the Kinnersley black hole, and show that both the shape and the Hawking temperature of the event horizon of Kinnersley black hole depend on not only the time, but also on the angle. The location and the temperature coincide with those obtained by investigating the Hawking effect of Klein-Gordon particles in the accelerating Kinnersley black hole. [5]

The metric of a non-uniformly rectilinearly accelerating Kinnersley black hole [9] is given in the advanced Eddington-Finkelstein coordinate system by

$$\begin{aligned} ds^2 &= 2dv(Gdv - dr - r^2 f d\theta) \\ &\quad - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \,, \end{aligned} \tag{1}$$

where $2G = 1 - 2M/r - 2ar\cos\theta - r^2f^2$, $f = -a\sin\theta$. The parameter $a = a(v)$ is the magnitude of acceleration, the mass $M(v)$ of the hole is a function of the advanced time v.

We choose such a complex null-tetrad in the Kinnersley black hole that its directional derivatives can be written as $D = -\partial_r$, $\Delta = \partial_v + G\partial_r$, $\delta = \frac{1}{\sqrt{2}r}\big(-r^2f\partial_r + \partial_\theta + \frac{i}{\sin\theta}\partial_\varphi\big)$, and $\overline{\delta}$ the complex conjugate of δ. It is not difficult to determine the non-vanishing Newman-Penrose complex spin coefficients [10] in the above null-tetrad as follows

$$\begin{aligned} &\rho = 1/r \,, \quad \tilde{\mu} = G/r \,, \quad \gamma = -G_{,r}/2 = -dG/2dr \,, \\ &\tau = -\tilde{\pi} = f/\sqrt{2} \,, \quad \beta = \cot\theta/(2\sqrt{2}r) \,, \quad \alpha = \tau - \beta \,, \\ &\nu = \big[(2rG - r^2G_{,r})f + r^2f_{,v} + G_{,\theta}\big]/(\sqrt{2}r) \,. \end{aligned} \tag{2}$$

If the back reaction is neglected, the dynamic behavior of spin-1/2 test particles in a fixed background space-time (1) which is of Petrov type-D is described by the spinor form of the four coupled Chandrasekhar-Dirac equations [7] in the Newman-Penrose formalism, [10] namely

$$\begin{aligned} &(D + \epsilon - \rho)F_1 + (\overline{\delta} + \tilde{\pi} - \alpha)F_2 = i\mu G_1/\sqrt{2} \,, \\ &(\Delta + \tilde{\mu} - \gamma)F_2 + (\delta + \beta - \tau)F_1 = i\mu G_2/\sqrt{2} \,, \\ &(D + \epsilon^* - \rho^*)G_2 - (\delta + \tilde{\pi}^* - \alpha^*)G_1 = i\mu F_2/\sqrt{2} \,, \\ &(\Delta + \tilde{\mu}^* - \gamma^*)G_1 - (\overline{\delta} + \beta^* - \tau^*)G_2 = i\mu F_1/\sqrt{2} \,, \end{aligned} \tag{3}$$

where μ is the mass of Dirac particles. Inserting for the appropriate spin coefficients into the above equations and making substitutions $P_1 = \sqrt{2}rF_1$, $P_2 = F_2$, $Q_1 = G_1$, $Q_2 = \sqrt{2}rG_2$, we obtain

$$\begin{aligned}
&-\mathcal{D}_0 P_1 + \big(\mathcal{L} - r^2 f\mathcal{D}_2\big)P_2 = i\mu r Q_1\,, \\
&2r^2\mathcal{B}P_2 + \big(\mathcal{L}^\dagger - r^2 f\mathcal{D}_0\big)P_1 = i\mu r Q_2\,, \\
&-\mathcal{D}_0 Q_2 - \big(\mathcal{L}^\dagger - r^2 f\mathcal{D}_2\big)Q_1 = i\mu r P_2\,, \\
&2r^2\mathcal{B}Q_1 - \big(\mathcal{L} - r^2 f\mathcal{D}_0\big)Q_2 = i\mu r P_1\,.
\end{aligned} \tag{4}$$

in which we have defined operators $\mathcal{B} = \partial_v + G\mathcal{D}_1 + G_{,r}/2$, $\mathcal{D}_n = \partial_r + n/r$, $\mathcal{L} = \partial_\theta + \frac{1}{2}\cot\theta - \frac{i}{\sin\theta}\partial_\varphi$, and $\mathcal{L}^\dagger = \partial_\theta + \frac{1}{2}\cot\theta + \frac{i}{\sin\theta}\partial_\varphi$.

Although Eq. (4) can not be decoupled, to investigate the Hawking radiation of spin-1/2 particles, one should concern about the behavior of Dirac spinor components near the event horizon only. Because the Chandrasekhar-Dirac equation (4) can be satisfied by identifying Q_1, Q_2 with $\bar{P}_2$, $-\bar{P}_1$, respectively, so we only need deal with a pair of components P_1, P_2. As the space-time is symmetric about φ-axis, we can introduce the following GTCT [4]

$$\begin{aligned}
&r_* = r + \frac{1}{2\kappa}\ln(r - r_H)\,, \\
&v_* = v - v_0\,, \quad \theta_* = \theta - \theta_0\,,
\end{aligned} \tag{5}$$

where $r_H = r_H(v, \theta)$ is the location of the event horizon, κ is an adjustable parameter and is unchanged under tortoise transformation. Both parameters v_0 and θ_0 are arbitrary constants.

Under the transformation (5), Eq. (4) with respect to (P_1, P_2) can be reduced to the following limiting form near the event horizon

$$\begin{aligned}
&2r_H^2\big[G(r_H) - r_{H,v}\big]\partial_{r_*}P_2 - \big(r_{H,\theta} + r_H^2 f_0\big)\partial_{r_*}P_1 = 0\,, \\
&\partial_{r_*}P_1 + \big(r_{H,\theta} + r_H^2 f_0\big)\partial_{r_*}P_2 = 0\,,
\end{aligned} \tag{6}$$

after being taken the $r \to r_H(v_0, \theta_0)$, $v \to v_0$ and $\theta \to \theta_0$ limits. We denote $f_0 = -a(v_0)\sin\theta_0$.

If the derivatives $\partial_{r_*}P_1$ and $\partial_{r_*}P_2$ in Eq. (6) are not equal to zero, the existence condition of nontrial solutions for P_1 and P_2 is that the determinant of Eq. (6) vanishes, which gives the following equation to determine the location of horizon

$$2G(r_H) - 2r_{H,v} + r_H^2 f_0^2 + 2f_0 r_{H,\theta} + r_{H,\theta}^2 r_H^{-2} = 0\,. \tag{7}$$

The location of the event horizon and the shape of the black hole change with time.

Now let us consider the asymptotic behaviors of the second-order form of Dirac equation near the event horizon. Given the GTCT in Eq. (5), the limiting form of the second-order equations for the two-component spinor (P_1, P_2), when r approaches $r_H(v_0, \theta_0)$, v goes to v_0 and θ goes to θ_0, reads

$$\begin{aligned}
\mathcal{K}P_1 &+ \big[-A + r_H^2 G_{,r}(r_H) + r_H^3 f_0^2 - r_H^2 f_0\cot\theta_0 \\
&- r_H^2 f_{0,\theta} - (r_H f_0 + \cot\theta_0)r_{H,\theta} - r_{H,\theta\theta}\big]\partial_{r_*}P_1 \\
&+ 2r_H^2\big\{r_H^2 f_{0,v} + G_{,\theta}(r_H) - G(r_H)r_{H,\theta}r_H^{-1} \\
&- r_H f_0[r_H G_{,r}(r_H) + r_{H,v} - 2G(r_H)]\big\}\partial_{r_*}P_2 = 0\,,
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\mathcal{K}P_2 &+ \big\{-A + 3r_H^2 G_{,r}(r_H) + 2r_H[2G(r_H) - r_{H,v}] \\
&+ 5r_H^3 f_0^2 - r_H^2 f_0\cot\theta_0 + (3f_0 r_H - \cot\theta_0)r_{H,\theta} \\
&- r_H^2 f_{0,\theta} - r_{H,\theta\theta}\big\}\partial_{r_*}P_2 + r_{H,\theta}r_H^{-1}\partial_{r_*}P_1 = 0\,.
\end{aligned} \tag{9}$$

where the operator $\mathcal{K}$ represents a term involving the second derivatives

$$\begin{aligned}
\mathcal{K} = \Big\{&\frac{A}{2\kappa} + 2r_H^2[2G(r_H) - r_{H,v}] + 2r_H^4 f_0^2 \\
&+ 2f_0 r_{H,\theta}r_H^2\Big\}\partial_{r_*}^2 + 2r_H^2\partial_{r_* v_*}^2 - 2(f_0 r_H^2 + r_{H,\theta})\partial_{r_*\theta_*}^2
\end{aligned}$$

With the aid of the event horizon equation (7), we know that the coefficient A is an infinite limit of 0/0-type. By means of the L' Hôspital rule, we get the following result

$$\begin{aligned}
A &= \lim_{r\to r_H}\frac{2r^2(G - r_{H,v}) + r^4 f^2 + 2fr^2 r_{H,\theta} + r_{H,\theta}^2}{r - r_H} \\
&= 2r_H^2 G_{,r}(r_H) + 2r_H^3 f_0^2 - 2r_{H,\theta}^2/r_H\,.
\end{aligned} \tag{10}$$

Now we select the adjustable temperature parameter κ in the operator $\mathcal{K}$ such that

$$\begin{aligned}
r_H^2 \equiv\ &\frac{A}{2\kappa} + 2r_H^2[2G(r_H) - r_{H,v}] + 2r_H^4 f_0^2 \\
&+ 2f_0 r_H^2 r_{H,\theta} = \frac{r_H^3 G_{,r}(r_H) + r_H^4 f_0^2 - r_{H,\theta}^2}{\kappa r_H} \\
&+ 2G(r_H)r_H^2 + r_H^4 f_0^2 - r_{H,\theta}^2\,,
\end{aligned} \tag{11}$$

which means the temperature of the horizon is

$$\kappa = \frac{r_H^2 G_{,r}(r_H) + r_H^3 f_0^2 - r_{H,\theta}^2/r_H}{r_H^2[1 - 2G(r_H)] - r_H^4 f_0^2 + r_{H,\theta}^2} \,. \tag{12}$$

With such a parameter adjustment and using Eq. (6), we can recast Eqs. (8,9) into an united standard wave equation near the event horizon

$$\big(\partial_{r_*}^2 + 2\partial_{r_* v_*}^2 - 2C_1\partial_{r_*\theta_*}^2 + 2C_2\partial_{r_*}\big)\Psi = 0 \,, \tag{13}$$

where $C_1 = f_0 + r_{H,\theta}/r_H^2$ and C_2 will be regarded as finite real constants,

$$\begin{aligned} 2C_2 &= 2G(r_H)r_H^{-1} - G_{,r}(r_H) - 2f_0\cot\theta_0 + 2r_{H,\theta}^2 r_H^{-3} \\ &\quad - (\cot\theta_0 r_{H,\theta} + r_{H,\theta\theta})r_H^{-2} + (f_0 + r_{H,\theta}r_H^{-2})\big\{r_H^2 f_{0,v} \\ &\quad + [2r_H G(r_H) - r_H^2 G_{,r}(r_H)]f_0 + G(r_H)r_{H,\theta} \\ &\quad - r_H G_{,\theta}(r_H)\big\}/[G(r_H) - r_{H,v}] \,, \quad \text{for } \Psi = P_1 \,, \\ 2C_2 &= 2G(r_H)r_H^{-1} + G_{,r}(r_H) + 2r_H f_0^2 - 2f_0\cot\theta_0 \\ &\quad - (\cot\theta_0 r_{H,\theta} + r_{H,\theta\theta})r_H^{-2} \,, \quad \text{for } \Psi = P_2 \,. \end{aligned}$$

Separating variables as $\Psi = R(r_*)\exp[\lambda\theta_* + i(m\varphi - \omega v_*)]$, one obtains

$$R'' = 2(i\omega - C_0)R' \,, \quad R = R_1 e^{2(i\omega - C_0)r_*} + R_0 \,, \tag{14}$$

where $C_0 = C_2 - \lambda C_1$, in which a real constant λ is introduced in the separation of variables.

The ingoing wave and the outgoing wave to Eq. (13) are

$$\Psi_{\rm in} \sim \exp[-i\omega v_* + im\varphi + \lambda\theta_*] \,, \tag{15}$$

$$\Psi_{\rm out} \sim \Psi_{\rm in} e^{2(i\omega - C_0)r_*} \,, \quad (r > r_H) \,. \tag{16}$$

The outgoing wave $\Psi_{\rm out}(r > r_H)$ is not analytic at the event horizon $r = r_H$, but can be analytically extended from the outside of the hole into the inside of the hole through the lower half complex r-plane by $(r - r_H) \to (r_H - r)e^{-i\pi}$ to

$$\widetilde{\Psi_{\rm out}} = \Psi_{\rm out} e^{i\pi C_0/\kappa} e^{\pi\omega/\kappa} \,, \quad (r < r_H) \,. \tag{17}$$

Following the method of Damour-Ruffini-Sannan's, [11] the relative scattering probability of the outgoing wave at the horizon and the thermal radiation spectrum of Dirac particles from the event horizon of the hole are

$$\left|\frac{\Psi_{\rm out}}{\widetilde{\Psi_{\rm out}}}\right|^2 = e^{-2\pi\omega/\kappa} \,, \tag{18}$$

$$\langle \mathcal{N}(\omega) \rangle \sim \frac{1}{e^{\omega/T_H} + 1} \,, \tag{19}$$

with the Hawking temperature being

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_H} \cdot \frac{M r_H - r_H^3 a\cos\theta_0 - r_{H,\theta}^2}{M r_H + r_H^3 a\cos\theta_0 + r_{H,\theta}^2/2} \,. \tag{20}$$

In conclusion, we have studied the Hawking radiation of Dirac particles in an arbitrarily accelerating Kinnersley black hole whose mass changes with time. Equations (7) and (12) give the location and the temperature of event horizon of the accelerating Kinnersley black hole, which depend not only on the advanced time v but also on the polar angle θ. Eq. (19) shows the thermal radiation spectrum of Dirac particles in an arbitrarily rectilinearly accelerating Kinnersley black hole. They are in accord with that derived from discussing on the thermal radiation of Klein-Gordon particles in the same space-time. [5]

S.Q. Wu is very grateful to Dr. Jeff Zhao at Motorola Company for his long-term help. This work was supported partially by the NSFC under Grant No.19875019.

[1] Hawking S W 1974 *Nature* **248** 30; 1975 *Commun. Math. Phys.* **43** 199

[2] Liu L and Xu D Y 1980 *Acta Phys. Sin.* **29** 1617; Zhao Z, Gui Y X and Liu L 1981 *Acta Astrophys. Sin.* **29** 141; Xu C M and Shen Y G 1982 *Acta Phys. Sin.* **31** 1035; Xu D Y 1983 *Acta Phys. Sin.* **32** 225 (All in Chinese)

[3] Wu S Q and Cai X 2000 *IL Nuovo Cimento* B **115** 143; 2000 *Int. J. Theor. Phys.* **39** 2215

[4] Zhao Z and Dai X X 1991 *Chin. Phys. Lett.* **8** 548

[5] Luo Z Q and Zhao Z 1993 *Acta Phys. Sin.* **42** 506 (in Chinese); Zhu J Y, Zhang J H and Zhao Z 1994 *Acta Astronomica Sinica* **35** 246 (in Chinese); Zhao Z, Zhang J H and Zhu J Y 1995 *Int. J. Theor. Phys.* **34** 2039

[6] Li Z H and Zhao Z 1993 *Chin. Phys. Lett.* **10** 126

[7] Chandrasekhar S 1983 *The Mathematical Theory of Black Holes* (New York: Oxford University Press)

[8] Wu S Q and Cai X 2001 *Chin. Phys. Lett.* **18** 485; 2001 *Gen. Rel. Grav.* **33** 1181

[9] Kinnersley W 1969 *Phys. Rev.* **186** 1335

[10] Newman E T and Penrose R 1962 *J. Math. Phys.* **3** 566

[11] Damour T and Ruffini R 1976 *Phys. Rev.* D **14** 332; Sannan S 1988 Gen. Rel. Grav. **20** 239